Deterministic spontaneous appearance of traffic jams in slightly inhomogeneous traffic flow

B. S. Kerner, P. Konhäuser, and M. Schilke
Research Institute, Daimler-Benz AG, F1V/VT, HPC G253, 70546 Stuttgart, Germany
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It is shown that a traffic jam can spontaneously be self-formed in a deterministic way in traffic flow, i.e., even if fluctuations in traffic flow are negligible and they need not be taken into account. This effect may be usual for a highway, where traffic flow can always be slightly inhomogeneous due to entering traffic to on-ramps and leaving traffic to off-ramps.

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I. INTRODUCTION

Experimental investigations of traffic flow have shown that traffic jams can appear there (e.g., [1]). It may be assumed that an appearance of a traffic jam without obvious reason could be a local effect. This supposition has recently been confirmed by the investigations of the local cluster effect: a self-formation of a local cluster of vehicles in the initially homogeneous traffic flow [2]. This localized structure can be formed in the initially homogeneous traffic flow, if the density exceeds a boundary (threshold) density and a localized fluctuation whose amplitude exceeds some critical value appears [2].

A homogeneous traffic flow is obviously a hypothetical state of traffic flow: A real traffic flow on a highway is always inhomogeneous, for example, due to entering traffic to on-ramps and leaving traffic to off-ramps. In this article, based on numerical investigations of the kinetic model of traffic flow, it will be shown that a process of a formation of a traffic jam shows qualitatively new peculiarities even in a slightly inhomogeneous traffic flow: A traffic jam on a highway can spontaneously appear in a *deterministic way,* i.e., even if fluctuations are negligible.

II. TRAFFIC JAMS IN INITIALLY INHOMOGENEOUS FLOW

A. Kinetic model of traffic flow on a highway

The kinetic model of traffic flow on a long road with I on-ramps and J off-ramps includes the continuity equation

$$\partial\rho/\partial t+\partial q/\partial x=\sum_{i=1}^{I} g_i(x-x_i,t)-\sum_{j=1}^{J} r_j(x-x_j,t) , \quad (1)$$

the equation of motion [3]

$$\rho[\partial v/\partial t+v\partial v/\partial x]=\rho[V(\rho)-v]/\tau-c_0^2\partial\rho/\partial x +\mu\partial^2 v/\partial x^2 , \quad (2)$$

and the boundary conditions

$$v(0,t)=v(L,t),\quad w(0,t)=w(L,t) ,$$
$$\rho(0,t)=\rho(L,t) . \quad (3)$$

In (1)–(3) $q(x,t)=\rho(x,t)v(x,t)$ is the flux, $\rho(x,t)$ is the density ($0<\rho\le\hat{\rho}$), and $v(x,t)$ is the average speed of vehicles ($v\ge 0$), $w(x,t)=\partial v/\partial x$, L is the length of the road, τ=const, μ=const, $\hat{\rho}$ is the maximal possible density, V is a safe ("maximal and out of danger") speed which is achieved in a both time-independent and homogeneous traffic flow, where the density will be designated as ρ_h, the average speed as v_h, $v_h=V(\rho_h)$, the flux as q_h, $q_h=\rho_h v_h$. $V(\rho)$ is a monotonous decreasing function of ρ, i.e., its derivative $dV(\rho)/d\rho\le 0$ [1,4,5]. The dependence $q_h=Q(\rho_h)$, where $Q(\rho)=\rho V(\rho)$, is called the fundamental diagram [1,4,5].

Equation (2) formally follows from the Navier-Stokes equations, to be more precisely, the "Navier-Stokes-like" equation for traffic flow [3],

$$\rho[\partial v/\partial t+v\partial v/\partial x]=X-\partial p/\partial x+(\partial/\partial x)(\mu\partial v/\partial x) . \quad (4)$$

Notice that in traffic flow the meaning of the terms on the right-hand side of Eq. (4) are completely different compared with classical physical systems. For example, in a gas, the pressure p as well as the viscosity μ appear due to the variance of the velocity distribution. In traffic flow, contrarily, the terms $\partial p/\partial x$ and $(\partial/\partial x)(\mu\partial v/\partial x)$ are even present when the variance of the speed distribution of vehicles is zero. These terms are present due to the perception, decision making and action of drivers in the case of an inhomogeneous flow. Therefore, they can be considered as some kind of anticipation factors. For example, the term $-\partial p/\partial x$ causes acceleration of drivers if the density ρ decreases and slowing down when the density increases (e.g., [5]). This means that the gradient of the "pressure" should have the same sign as the gradient of the density: $\partial p/\partial x=c_0^2\partial\rho/\partial x$, where $c_0^2>0$. The value c_0^2 may be, however, in two limit ranges of the density nearly equal zero: (i) when the density everywhere on a road is so low that all drivers move practically with their "free speed" v_f and do not react on a change in density and (ii) when the average speed due to very high density is nearly zero and drivers cannot move even if $\partial p/\partial x\neq 0$. A variety of functions $c_0^2(\rho)$ that fulfills the conditions (i) and (ii) can be important in some cases. However, for the investigations of traffic jams made in [2,3] and presented here one can use in Eq. (2) c_0^2=const for the following reasons: (a) The range of density (i) is not relevant for $\rho\ge\rho_b$ [2], where traffic jams can be formed. (b) The range of density (ii) is usually realized inside traffic jams, but there we have $\partial\rho/\partial x=0$. (c) In an intermediate range of density a possible dependence $c_0^2(\rho)$ is not important for qualitative results. The value c_0 may be considered as the velocity of the propagation of a possible or of a necessary reaction (i.e., a change in acceleration or in slowing down) of drivers which is caused by some local change in traffic flow downstream: The delay time τ_r of this reaction of the drivers on the local change in traffic flow which is situated at a distance Δx downstream is $\tau_r \approx \Delta x / c_0$.

The terms on the right-hand side of Eq. (1) include entering traffic g_i to the on-ramp "i" situated at $x = x_i$ and exiting traffic r_j from the highway to the off-ramp "j" being situated at $x = x_j$. For a road long enough, the length of the section on which traffic is entering or exiting the road may be considered as negligible short. Therefore, it can be assumed that $g_i(x-x_i,t) = g_i^0(t)\varphi_i(x-x_i)$ and $r_j(x-x_j,t) = r_j^0(t)\phi_j(x-x_j)$, where functions $\varphi_i(x-x_i)$ and $\phi_j(x-x_j)$ should be strongly localized near the corresponding on-ramp or off-ramp. $q_a^{(i)} = g_i^0(t)\int_0^L \varphi_i(x-x_i)dx$ or $q_d^{(j)} = r_j^0(t)\int_0^L \phi_j(x-x_j)dx$ are fluxes from on-ramps or through off-ramps. For the numerical investigations of the effect of a deterministic formation of traffic jams, the problem (1)–(3) has been solved. The algorithm of the numerical solution of such a problem has been described in [3].

B. Deterministic self-formation of traffic jams

The deterministic spontaneous appearance of traffic jams is linked with the "local breakdown" effect that starts in transition layers between different levels of the density in an inhomogeneous traffic flow. We will distinguish between two different cases: (i) motionless transition layers and (ii) moving transition layers. Conditions of an occurrence of the local breakdown effect in these two cases are essentially different.

1. Local breakdown effect in motionless transition layers of inhomogeneous traffic flow

Let us first propose that at time $t<0$ there is an initially homogeneous traffic flow with some density ρ_h, average speed of vehicles v_h, and the flux $q_h = v_h \rho_h$. Obviously that when, beginning at time $t=0$, some additional flux $q_a = q_a^{(1)}$ of vehicles appears from an on-ramp, traffic flow becomes inhomogeneous: The total flux $q_t = q_h + q_a$ causes traffic flow with higher density $\rho_{h1} > \rho_h$ directly downstream from the on-ramp [Fig. 1(a)]. If the total flux q_t is held constant, the transition layer, which appears between the initial value of density ρ_h and the new higher value of density ρ_{h1}, does not move [Fig. 1(a)]. When the region of traffic flow, which is formed by the flux q_t, is wide enough, the density ρ_{h1} corresponds nearly to a point on the fundamental diagram. When the flux q_a is slightly increased, the density ρ_{h1} is slightly increased, too: It will correspond to a higher point on the fundamental diagram. This gradual behavior of a change of the density holds as long as the total flux q_t is less than the value Q_{max} which corresponds to the maximum point (Q_{max}, ρ_0) on the fundamental diagram [Fig. 1(c)]. When the flux q_a is increased further, so that the total flux $q_t = q_h + q_a > Q_{max}$, a local avalanchelike change occurs [Fig. 1(b)]: The density increases and the average speed of vehicles decreases in an avalanchelike manner in the localized region that is situated downstream from the on-ramp.

Due to such a local breakdown effect, a traffic jam is formed. The form and the properties of this traffic jam [Fig. 1(b)] are similar to those found in [2] when a traffic jam due to an occurrence of a localized fluctuation appeared. On the contrary, the considered local breakdown effect results in a dynamical local restructuring, which is not associated with the presence of fluctuations. Fluctuations only make the local breakdown possible even before the flux q_t reaches the critical value Q_{max}, which defines the local breakdown threshold without taking fluctuations into account [6].

To understand the reasons of the local breakdown

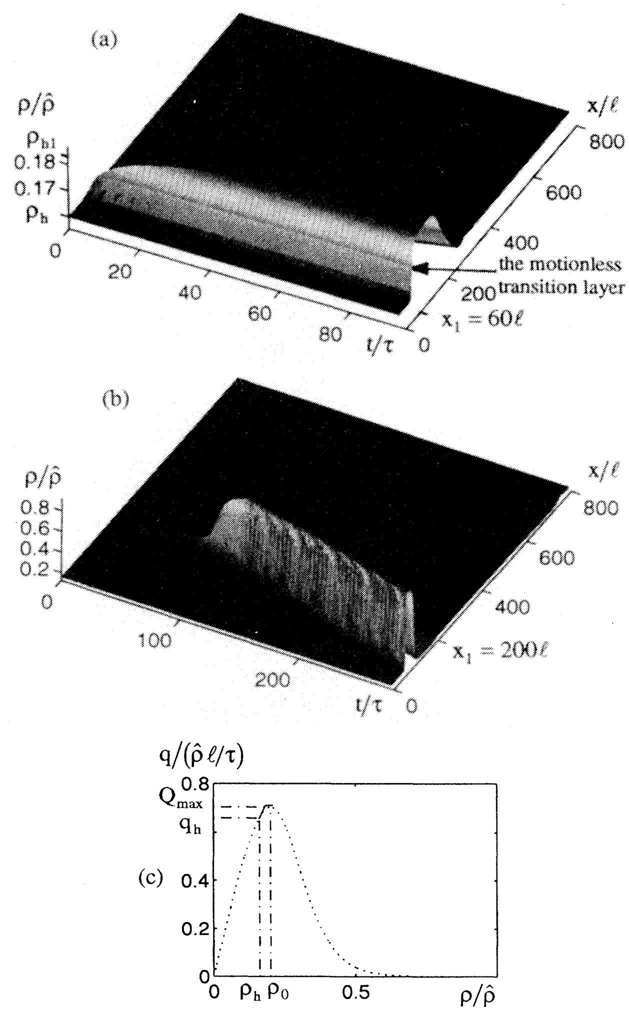

FIG. 1. A deterministic formation of a traffic jam in motionless transition layers: (a) the dependence $\rho(x,t)$ for the case, when a traffic jam is not formed because $q_t < Q_{max}$ ($q_t = 0.696\hat{\rho}l/\tau$, $\rho_{h1} = 0.1833\hat{\rho}$, $Q_{max} = 0.7035\hat{\rho}l/\tau$, $\rho_0 = 0.199\hat{\rho}$); (b) $\rho(x,t)$ for the case, when a traffic jam is formed due to $q_t > Q_{max}$($q_t = 0.7111\hat{\rho}l/\tau$); (c) the fundamental diagram (dotted line) and the true function $q(\rho)$ (solid line), corresponding to the transition layer at $t=69\tau$, for the case (b). Results of the numerical investigations for $c_0 = 2.0497 l/\tau$, $V(\rho) = 5.0461[(1+\exp\{[(\rho/\hat{\rho})-0.25]/0.06\})^{-1} - 3.72\times 10^{-6}]l/\tau$ [2,3], $L=800l$, $\rho_h = 0.16\hat{\rho}$ ($q_h = 0.6601\hat{\rho}l/\tau$), the on-ramps are situated at $x_1 = 60l$ for (a) and $x_1 = 200l$ for (b) and (c). $l = \sqrt{\mu\hat{\rho}^{-1}}\tau$.

effect, observe that the function $\rho_h(q_h)$, is S shaped, and at $q_h = Q_{\max}$ we have $d\rho/dq = \infty$ [dotted line in Fig. 1(c)]. Recall that the function $\rho_h(q_h)$ nearly defines the relation between values ρ_{h1} and q_t directly downstream from the on-ramp. When $q_t = q_{h1} + q_a \to Q_{\max}$, the true function $\rho_{h1}(q_t)$ in the transition layer downstream from the on-ramp abuts on this limiting point $q_h = Q_{\max}$. Therefore, for $q_t > Q_{\max}$ [solid line in Fig. 1(c)] the density should increase abruptly downstream from the on-ramp from $\rho_{h1} \approx \rho_0$ [Fig. 1(c)]. This local breakdown naturally results in an appearance of a traffic jam [Fig. 1(b)].

2. *Local breakdown effect in moving transition layers of inhomogeneous traffic flow*

When the flux is higher than $Q_{\max}$ [solid line in Fig. 1(c)], it is quite expected that some kind of "catastrophic" processes will appear in the traffic flow. However, a deterministic occurrence of a traffic jam can usually be realized on a highway, even if the flux in the traffic flow in any region of the road is considerably *lower* than $Q_{\max}$.

To show this effect, let us return to the case considered in Fig. 1(a), where the total flux $q_t < Q_{\max}$ and, therefore, the local breakdown effect has not occurred. Because in Fig. 1(a) during the time $0 \le t \le 95\tau$ the flux q_a from the on-ramp was constant, the transition layer could not move. In real traffic flow, a flux q_a is not usually a constant during a long period of time. For example let us consider the situation, when

$$q_a = \begin{cases} q_a, & 0 \le t \le 95\tau \\ 0, & t > 95\tau \end{cases}$$

[Fig. 2(a)]. In this case, the transition layer that has been caused by the entering traffic to the on-ramp during the time $0 \le t \le 95\tau$ [Fig. 1(a)], at $t > 95\tau$ begins to move [Fig. 2(a)] with a positive velocity,

$$v_{\mathrm{tr}} \approx (q_t - q_h)/(\rho_{h1} - \rho_h) = q_a/(\rho_{h1} - \rho_h) . \tag{5}$$

If the difference $\rho_{h1} - \rho_h$ in the transition layer is small enough, it follows from (5) that for the velocity v_{tr} we can write the formula,

$$v_{\mathrm{tr}} \approx v_h + \rho_h dV/d\rho|_{\rho=\rho_h} , \tag{6}$$

where it has been taken into account that $q_t \cong v_{h1}\rho_{h1}$ and $q_h = v_h \rho_h$. Although the total flux $q_t < Q_{\max}$, a self-formation of a traffic jam occurs after the transition layer starts to move (Fig. 2).

Let us discuss necessary conditions of such a deterministic appearance of a traffic jam (Figs. 2(a) and 2(c)]. First notice that directly downstream from the moving transition layer a deterministic local perturbation of the density of vehicle occurs [Fig. 2(b)]. Vehicles that reach the moving transition layer with higher speed v_h must slow down inside the transition layer, where the density is increasing. To satisfy the safety requirements, drivers arriving at the transition layer slow down more than they would make in a "quasistationary" case, i.e., when the density were only smoothly changing in space. Such a local deterministic perturbation [Fig. 2(b)] can only be seen directly downstream the transition layer, if a velocity v_q of this perturbation is nearly equal to the velocity of the transition layer v_{tr}, i.e., when

$$v_{\mathrm{tr}} \approx v_q . \tag{7}$$

Indeed, if, for example, v_{tr} were considerably more than v_q, the local perturbation could not be localized directly downstream the transition layer and, therefore, the local breakdown effect could not start. The velocity v_q of small amplitude perturbations in traffic flow can be estimated from the formula (e.g., [2,3])

$$v_q \approx v_h - c_0 . \tag{8}$$

At low density, when the derivative $|dV/d\rho|$ is very small, we have from (6) $v_{\mathrm{tr}} \approx v_h$, i.e., the velocity v_{tr} is considerably more than v_q (8). Therefore, at low density the condition (7) cannot be fulfilled. When the density is increased, the derivative $|dV/d\rho|$ increases, too (e.g., [3,4]). Consequently, the velocity v_{tr} (6) decreases and it can approach the velocity v_q (8). Therefore, there should be the density for which condition (7) is fulfilled. The formula for an estimation of this density can be found from (7), if (6) and (8) are taken into account,

$$-(\rho_h/c_0)dV/d\rho|_{\rho=\rho_h} \approx 1 . \tag{9}$$

The density that follows from (9) approximately coincides, when L is large enough, with the characteristic value ρ_{c1} (or ρ_{c2}) (see the formulas (10) in [3]). For a starting of the local breakdown effect, besides (9), it is also necessary that an amplitude of the deterministic perturbation exceeds some critical value. The amplitude of this perturbation is increased if either the density ρ_h or

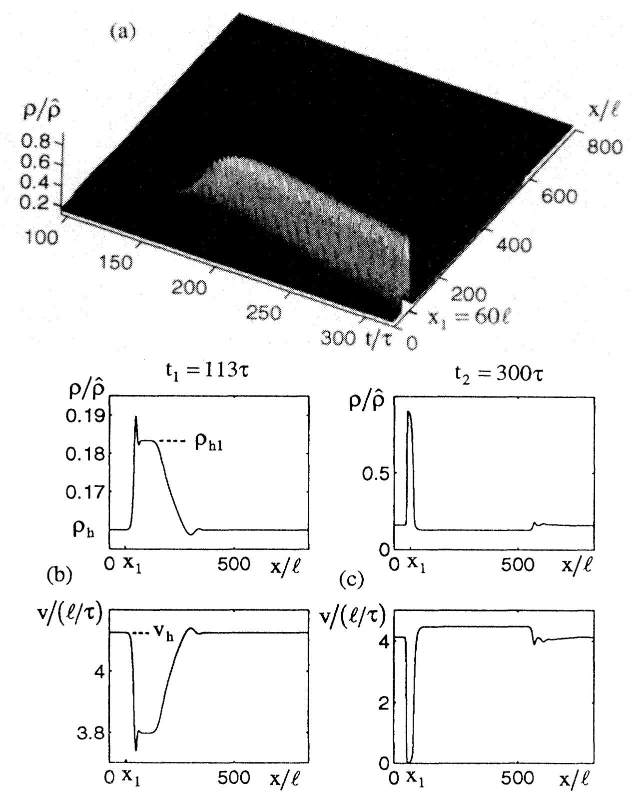

FIG. 2. A deterministic formation of a traffic jam in a moving transition layer: (a) the dependence $\rho(x,t)$ for the time $t > 95\tau$, when $q_a = 0$ and therefore the transition layer shown in Fig. 1(a) begins to move; (b) and (c) the corresponding distributions $\rho(x)$ and $v(x)$ in two intermediate moments of time $t_1 = 113\tau$ (b) and $t_2 = 300\tau$ (c). The other parameters are the same as in Fig. 1(a).

the density ρ_{h1} are increased. For this reason the occurrence of the local breakdown effect depends both on the density ρ_h and on ρ_{h1}. If ρ_h is considerably lower than ρ_{c1}, to start the local breakdown effect, the density ρ_{h1} should be higher than ρ_{c1}, but this density ρ_{h1} can be considerably lower than the value ρ_0 [Fig. 1(c)]. The density ρ_h can be lower than ρ_{c1}, but it should be higher than the boundary (threshold) density ρ_b [2].

The local breakdown effect and a deterministic formation of traffic jams and of other structures in slightly inhomogeneous traffic flow can be realized not only in moving transition layers from "low to high" density (Fig. 2), but also in any other moving transition layers, for example, in moving transition layers from "high to low" density and in a dense flow $\rho'_{\rm cr} < \rho \le \rho_{c2}$, (Fig. 3), when transition layers move with a negative velocity.

III. CONCLUSIONS AND DISCUSSIONS

(i) In a slightly inhomogeneous traffic flow traffic jams can spontaneously be formed in a deterministic way, i.e., even if fluctuations in traffic flow are negligible. Such deterministic self-formation of traffic jams is caused by the local breakdown effect, which results in a dynamical local restructuring. The local breakdown can start in a vicinity of moving transition layers between different levels of the density in the traffic flow.

(ii) Summarizing the results of [2,3] and this article, one can divide the whole possible range of density into four regions (Fig. 3): (a) a stable state of a traffic flow, where any short-time perturbations fade in time, (b) a metastable state, where traffic jams, including complex localized structures of a lot of traffic jams, can appear, if localized fluctuations occur whose amplitudes exceed the critical value [2], (c) an unstable state, where complex nonhomogeneous nonstationary structures that cover the whole road are formed [2], and (d) a metastable state, where anticlusters [2] and local "dipole layers" [7] can be formed, if localized fluctuations occur whose amplitudes exceed the critical value. However, in the ranges $\rho_{c1} \le \rho < \rho_{\rm cr}$ and $\rho'_{\rm cr} < \rho \le \rho_{c2}$ (Fig. 3) traffic flow only formally may be considered as the metastable states. Indeed, all mentioned types of structures in the range of the density $\rho_{c1} \le \rho \le \rho_{c2}$ (Fig. 3) can spontaneously appear in a deterministic way, i.e., even if fluctuations in traffic flow may be negligible.

(iii) Besides macroscopic (hydrodynamic) traffic flow models, there are also microscopic models, where the behavior of each individual vehicle is taken into account. In particular, the microscopic approach to traffic flow based on cellular automation models has recently been developed (e.g., [8–10]). In this approach, one-dimensional [8,10] and also two-dimensional models [9] of traffic flow have been proposed and investigated. As well as the investigations of the macroscopic model [3], cellular automation models of traffic flow show a transition from an initially homogeneous to a jammed state. The recent investigations [10] of the model proposed by Nagel and Schreckenberg [8] are in good qualitative agreement with our previous conclusions [2] about processes of appearance and disappearance of complex sequences of traffic jams.

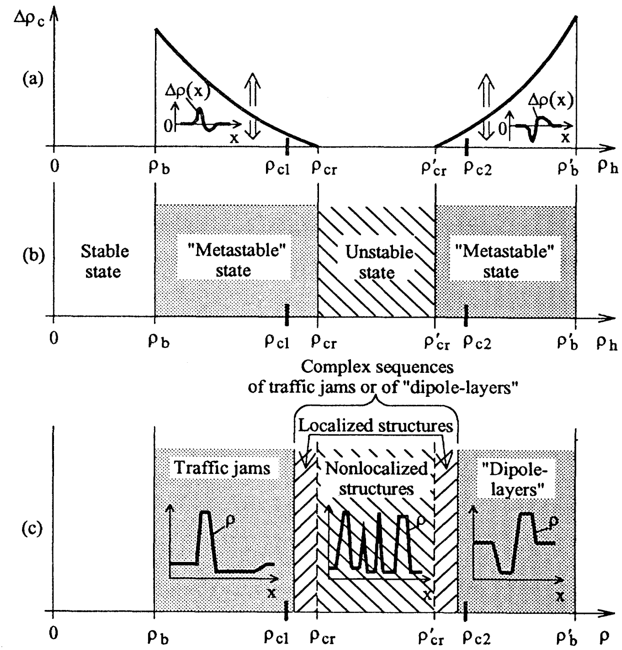

FIG. 3. Diagram of states of traffic flow: (a) dependencies of the critical amplitude $\Delta\rho_c$ of localized functions $\Delta\rho(x)$ on the density; (b) states of homogeneous traffic flow (with respect to the growth of localized fluctuations); (c) types of structures in traffic flow, both initially homogeneous and inhomogeneous. $\rho'_b \cong \rho^m_{\rm max}$. The sense of the designations ρ_b, ρ_{c1}, $\rho_{\rm cr}$, ρ_{c2} and $\rho^m_{\rm max}$ have been explained in [2]. In the range of density $\rho_{\rm cr} \le \rho \le \rho'_{\rm cr}$ localized perturbations of any amplitude grow.

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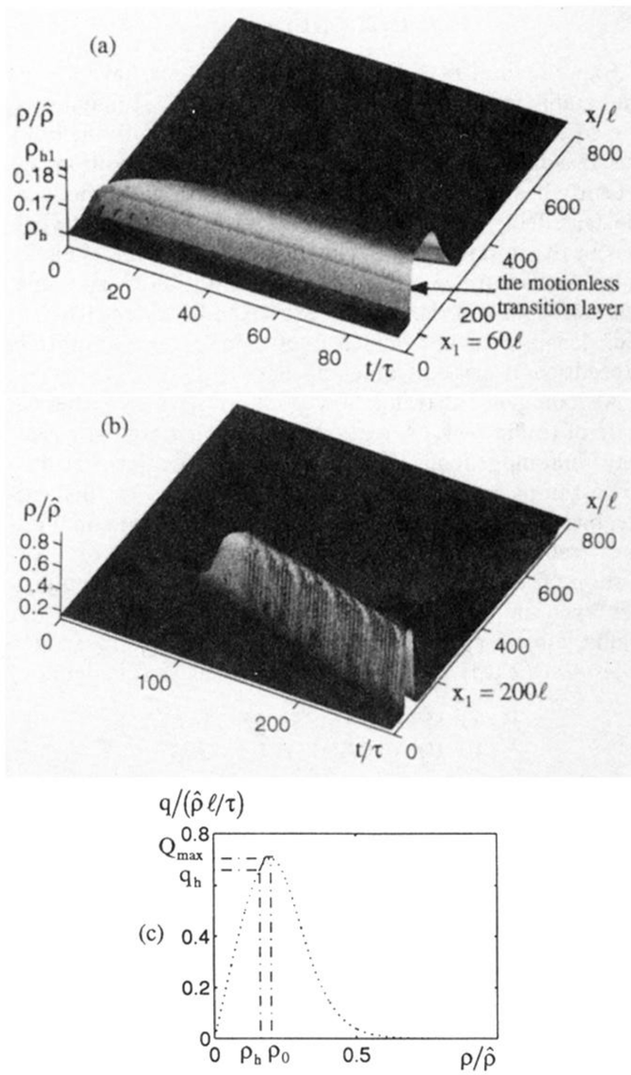

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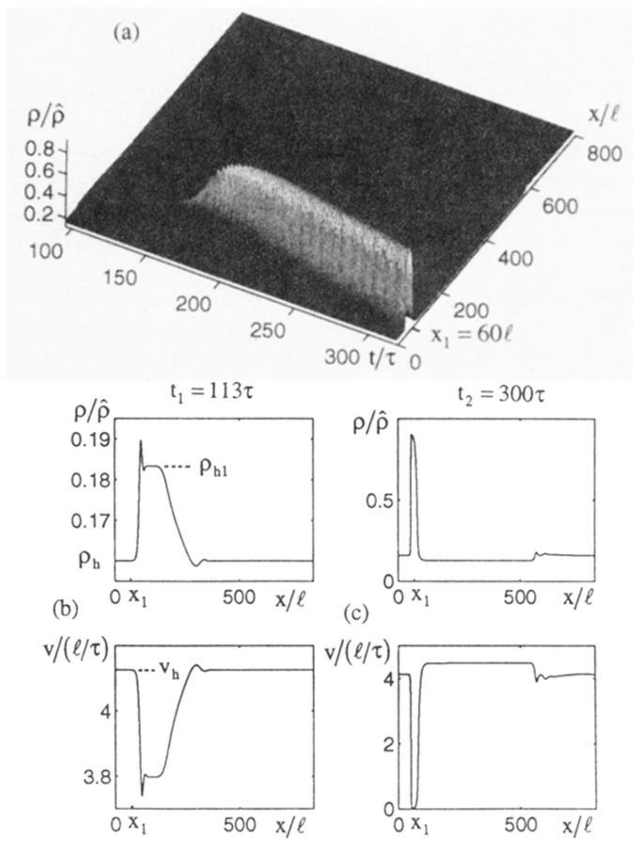

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